

# Flux dualization in broken $SU(2)$

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**ABSTRACT:** An  $SU(2)$  gauge theory is broken to  $U(1)$  by an adjoint scalar to produce magnetic monopoles. At a lower scale, this  $U(1)$  is further broken by a fundamental scalar to produce tubes of magnetic flux. We dualize the resulting theory to write an effective theory in terms of the macroscopic string variables. The monopoles are attached to the ends of the strings, and the flux is confined in the tubes.

## 1. Introduction:

It is widely believed that color confinement in the strong coupling regime should be a phenomenon dual to monopole confinement in a color superconductor at weak coupling. In this picture, the QCD vacuum behaves like a dual superconductor, created by condensation of magnetic monopoles, in which confinement is analogous to a dual Meissner effect. Quarks are then bound to the ends of a flux string [1, 2, 3] analogous to the Abrikosov-Nielsen-Olesen vortex string of Abelian gauge theory [4, 5].

A construction of flux strings in the Weinberg-Salam theory was suggested by Nambu [6], in which a pair of magnetic monopoles are bound by a flux string of Z condensate. The magnetic monopoles are introduced by hand. If we demand that the magnetic monopoles should appear from the underlying gauge theory, we need an additional adjoint scalar field. Such a construction of flux string, involving two adjoint scalar fields in an  $SU(2)$  gauge theory, has been discussed in [5, 7]. Recently there has been a resurgence of interest in such constructions [8, 9, 10, 11]. We have previously shown explicitly that [12] an  $SU(2)$  gauge theory broken by two adjoint scalar fields at different energy scales has configurations of magnetic monopoles bound by flux strings.

In this paper we consider an  $SU(2)$  gauge theory coupled to an adjoint scalar field as well as a fundamental scalar field. The two fields break the symmetry at two scales. At the higher scale the adjoint scalar breaks the symmetry down to  $U(1)$  and produces 't Hooft-Polyakov magnetic monopoles [13, 14, 15]. The fundamental scalar breaks the remaining  $U(1)$  symmetry at a lower scale and produces a flux string.

Our starting point is the Lagrangian

$$L = -\frac{1}{2}\text{Tr} (G_{\mu\nu}G^{\mu\nu}) + \text{Tr} (D_\mu\phi D^\mu\phi) + \frac{1}{2}(D_\mu\psi^\dagger)(D^\mu\psi) + V(\phi, \psi). \quad (1.1)$$

Here  $\phi$  is in the adjoint representation of  $SU(2)$ ,  $\phi = \phi^i\tau^i$  with real  $\phi^i$  and  $\psi$  is a fundamental doublet of  $SU(2)$ , with  $V(\phi, \psi)$  some interaction potential for the scalars. The  $SU(2)$  generators  $\tau^i$  satisfy  $\text{Tr} (\tau^i\tau^j) = \frac{1}{2}\delta^{ij}$ . The covariant derivative  $D_\mu$  and the Yang-Mills field strength tensor  $G_{\mu\nu}$  are defined as

$$(D_\mu\phi)^i = \partial_\mu\phi^i + g\epsilon^{ijk}A_\mu^j\phi^k, \quad (1.2)$$

$$G_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\epsilon^{ijk}A_\mu^jA_\nu^k, \quad (1.3)$$

$$(D_\mu\psi)_\alpha = \partial_\mu\psi_\alpha - igA_\mu^i\tau_{\alpha\beta}^i\psi_\beta. \quad (1.4)$$

We will sometimes employ vector notation, in which

$$D_\mu\vec{\phi} = \partial_\mu\vec{\phi} + g\vec{A} \times \vec{\phi}, \quad (1.5)$$

$$D_\mu\psi = \partial_\mu\psi - igA_\mu\psi, \quad (1.6)$$

$$\vec{G}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu, \text{ etc.} \quad (1.7)$$

Obviously,  $\vec{\phi}$  and  $\phi$  represent the same object. The simplest form of the potential  $V(\phi, \psi)$  that will serve our purpose is,

$$V(\phi, \psi) = -\frac{\lambda_1}{4}(|\phi|^2 - v_1^2)^2 - \frac{\lambda_2}{4}(\psi^\dagger\psi - v_2^2)^2 - V_{mix}(\phi, \psi). \quad (1.8)$$

Here  $v_1, v_2$  are the parameters of dimension of mass and  $\lambda_1, \lambda_2$  are dimensionless coupling constants. The last term  $V_{mix}(\phi, \psi)$  includes all mixing terms in the potential, which involve products of the two scalar fields in some way. We will take  $V_{mix}(\phi, \psi) = 0$  for now, so  $v_1$  and  $v_2$  are the local minima of the potential, and we will refer to them as the vacuum expectation values of  $\phi$  and  $\psi$ .

The adjoint scalar  $\phi$  acquires a vacuum expectation value (vev)  $\vec{v}_1$  which is a vector in internal space, and breaks the symmetry group down to  $U(1)$ . The 't Hooft-Polyakov monopoles are associated with this breaking. The other scalar field  $\psi$  also has a non-vanishing vev  $v_2$  which is a vector in the fundamental representation. This vector can be associated uniquely with a vector in the adjoint space which is free to wind around  $\vec{v}_1$ . A circle in space is mapped to this winding, giving rise to the vortex string. We then dualize the fields as in [16, 17, 18, 19, 20] to write the action in terms of string variables.

The idea of two-scale symmetry breaking in  $SU(2)$ , the first to produce monopoles and the second to produce strings, has appeared earlier [21]. Later this idea was used in a supersymmetric setting in [22, 23, 24], where the idea of flux matching, following Nambu [6] was also included. The model we discuss in this paper, with one adjoint and one fundamental scalar, has been considered previously in [10]. Here we construct the flux strings explicitly in non-supersymmetric  $SU(2)$  theory with 't Hooft-Polyakov monopoles of the same theory attached to the ends. The internal direction of symmetry breaking is left arbitrary, so that the magnetic flux may be chosen to be along any direction in the internal space. We also dualize the variables to write the effective theory of macroscopic string variables coupled to an antisymmetric tensor, and thus show explicitly that the flux at each end of the string is saturated by the magnetic monopoles, indicating confinement of magnetic flux.

## 2. Magnetic monopoles

We assume that  $v_1$ , the vacuum expectation value of  $\phi$ , is large compared to the energy scale we are interested in. Below the scale  $v_1$ , we find the  $\phi$  vacuum, defined by the equations

$$\begin{aligned} D_\mu \vec{\phi} &= 0, \\ |\phi|^2 &= v_1^2. \end{aligned} \tag{2.1}$$

Below  $v_1$ , the original  $SU(2)$  symmetry of the theory is broken down to  $U(1)$ . At low energies the theory is essentially Abelian, with the component of  $A$  along  $\phi$  remaining massless. We can now write the gauge field below the scale  $v_1$  as

$$\vec{A}_\mu = B_\mu \hat{\phi}_1 - \frac{1}{g} \hat{\phi}_1 \times \partial_\mu \hat{\phi}_1, \tag{2.2}$$

where  $B_\mu = \vec{A}_\mu \cdot \hat{\phi}_1$  and  $\hat{\phi}_1 = \vec{\phi}_1/v_1$  [25]. In this vacuum, until we include the second symmetry breaking,  $B_\mu$  is a massless mode. The other two components of  $A$ , which we call  $A^\pm$ , and the modulus of the scalar field  $\phi$  acquire masses,

$$M_{A^\pm} = gv_1, \quad M_{|\phi|} = \sqrt{\lambda_1} v_1. \tag{2.3}$$

Well below  $v_1$  the modes  $A^\pm$  are not excited, so they will not appear in the low energy theory. The second term on the right hand side of Eq. (2.2) corresponds to the gauge field for SU(2) magnetic monopoles [11].

A straightforward calculation shows that,

$$\text{Tr}(G_{\mu\nu}G^{\mu\nu}) = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}, \quad (2.4)$$

where

$$F_{\mu\nu} = \partial_{[\mu}B_{\nu]} - \frac{1}{g}\hat{\phi}_1 \cdot \partial_\mu\hat{\phi}_1 \times \partial_\nu\hat{\phi}_1 \equiv \partial_{[\mu}B_{\nu]} + M_{\mu\nu}. \quad (2.5)$$

Then the Lagrangian can be written in the  $\phi$ -vacuum as

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\psi^\dagger)(D^\mu\psi) - \frac{\lambda_2}{4}(\psi^\dagger\psi - v_2^2)^2. \quad (2.6)$$

The second term of Eq. (2.5) is the ‘monopole term’. In a configuration where the scalar field at spatial infinity goes as  $\phi_1^i \rightarrow v_1 \frac{r^i}{r}$ , the  $(ij)^{th}$  component of the second term of Eq. (2.5) becomes  $-\frac{\epsilon_{ijk}r^k}{gr^3}$ , which we can easily identify as the field of a magnetic monopole. The flux for this monopole field is  $\frac{4\pi}{g}$ . On the other hand, a monopole with magnetic charge  $Q_m$  produces a flux of  $4\pi Q_m$ , and thus we find the quantization condition for unit charge,  $Q_m g = 1$ .

The scalar field  $\phi$  can be written as  $\phi(x) = |\phi(x)|\hat{\phi}(x)$ , where  $\hat{\phi}$  contains two independent fields (and  $x \equiv \vec{x}$ ). So under a gauge transformation  $\hat{\phi}$  has a trajectory on  $S^2$ . Since  $\phi$  is in the adjoint of SU(2), we can always write  $\phi$  as

$$\phi(x) = |\phi(x)|g(x)\tau^3g^{-1}(x) = |\phi(x)|\hat{\phi}(x), \quad (2.7)$$

with  $g(x) \in \text{SU}(2)$ . Then for a given  $\phi(x)$ , we can locally decompose  $g(x)$  as  $g(x) = h(x)U(x)$ , with  $h(x) = \exp(-i\xi(x)\hat{\phi}(x))$ , and we can write

$$\phi(x) = |\phi(x)|U(\varphi(x), \theta(x))\tau^3U^\dagger(\varphi(x), \theta(x)), \quad (2.8)$$

Here  $\xi(x), \varphi(x), \theta(x)$  are angles on  $S^3 = \text{SU}(2)$ . The matrix  $U$  rotates  $\hat{\phi}(x)$  in the internal space, and is an element of  $\text{SU}(2)/\text{U}(1)$ , where the  $\text{U}(1)$  is the one generated by  $h$ . If  $|\phi|$  is zero at the origin and  $|\phi|$  goes smoothly to its vacuum value  $v_1$  on the sphere at infinity, the field  $\phi$  defines a map from space to the vacuum manifold such that second homotopy group of the mapping is  $\mathbb{Z}$ . Equating  $\phi$  with the unit radius vector of a sphere we can solve for  $U(\theta(x), \varphi(x))$ ,

$$U = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} e^{-i\varphi} \\ \sin \frac{\theta}{2} e^{i\varphi} & \cos \frac{\theta}{2} \end{pmatrix}. \quad (2.9)$$

An ’t Hooft-Polyakov monopole (in the point approximation, or as seen from infinity) at the origin is described by

$$U = \cos \frac{\theta}{2} \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} + \sin \frac{\theta}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (2.10)$$

where  $0 \leq \theta(\vec{x}) \leq \pi$  and  $0 \leq \varphi(\vec{x}) \leq 2\pi$  are two parameters on the group manifold. This choice of  $U(\vec{x})$  is different from that in Eq. (2.9) by a rotation of the axes. Both choices lead to the field configuration

$$\vec{\phi} = v_1 \frac{r^i}{r} \tau_i. \quad (2.11)$$

For this case,  $Q_m g = 1$ , as we mentioned earlier. A monopole of charge  $n/g$  is obtained by making the replacement  $\varphi \rightarrow n\varphi$  in Eq.s (2.9, 2.10). The integer  $n$  labels the homotopy class,  $\pi_2(SU(2)/U(1)) \sim \pi_2(S^2) \sim Z$ , of the scalar field configuration. Other choices of  $U(\vec{x})$  can give other configurations. For example, a monopole-anti-monopole configuration [26] is given by the choice

$$U = \sin \frac{(\theta_1 - \theta_2)}{2} \begin{pmatrix} 0 & -e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix} + \cos \frac{(\theta_1 - \theta_2)}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.12)$$

For our purposes, we will need to consider a  $\phi_1$ -vacuum configuration with  $U(\vec{x}) \in SU(2)$  corresponding to a monopole-anti-monopole pair separated from each other by a distance  $> 1/v_1$ . Then the total magnetic charge vanishes, but each monopole (or anti-monopole) can be treated as a point particle.

### 3. Flux tubes

We started with a theory with  $SU(2)$  symmetry and a pair of scalars  $\phi, \psi$ . The non zero vacuum expectation value  $v_1$  of the field  $\phi$  breaks the symmetry to  $U(1)$ , so that below  $v_1$  we have an effective Abelian theory with magnetic monopoles. The gauge group  $SU(2)$  acts transitively on the vacuum manifold  $S^2$ , so the Abelian effective theory is independent of the internal direction of  $\phi$ . The remaining symmetry of the theory is the  $U(1)$ , the little group of invariance of  $\phi$  on the vacuum manifold. This is the group of rotations around any point on the vacuum manifold  $S^2$ .

There is another scalar field  $\psi$  in the theory, a scalar in the fundamental representation of  $SU(2)$ . After breaking the original  $SU(2)$  down to the  $\phi$ -vacuum, the only remaining gauge symmetry of the  $SU(2)$  doublet  $\psi$  is a transformation by the little group  $U(1)$ . We will find flux tubes when this  $U(1)$  symmetry is spontaneously broken down to nothing. The elements of this  $U(1)$  are  $h(x) = \exp[i\xi(x)\hat{\phi}(x)]$ , rotations by an angle  $\xi(x)$  around the direction of  $\phi(x)$  at any point in space. This  $U(1)$  will be broken by the vacuum configuration of  $\psi$ .

Let us then define the  $\psi$ -vacuum by,

$$\psi^{*i} \psi^i = v_2^2 \quad (3.1)$$

$$D_\mu \psi = 0, \quad (3.2)$$

where  $D_\mu$  is defined using  $A_\mu$  in the  $\phi$ -vacuum, as in Eq. (2.2). Multiplying Eq. (3.2) by  $\psi^\dagger \hat{\phi}$  from the left, its adjoint by  $\hat{\phi} \psi$  from the right, and adding the results, we get

$$\begin{aligned} 0 &= \psi^\dagger \hat{\phi} D_\mu \psi + (D_\mu \psi^\dagger) \hat{\phi} \psi \\ &= \partial_\mu \left[ \psi^\dagger \hat{\phi} \psi \right], \end{aligned} \quad (3.3)$$

from which it follows that

$$\psi^\dagger \hat{\phi} \psi = \text{constant} , \quad (3.4)$$

or explicitly in terms of the components,

$$\text{Tr} \left[ \psi^\dagger_i \sigma_{ij}^\alpha \psi_j \tau_\alpha \hat{\phi} \right] = \text{constant} . \quad (3.5)$$

It follows that the components parallel and orthogonal to  $\phi$  are both constants. Then we can decompose

$$\psi^\dagger_i \sigma_{ij}^\alpha \psi_j \tau_\alpha = v_2^2 \cos \theta_c \hat{\phi} + v_2^2 \sin \theta_c \hat{\kappa} , \quad (3.6)$$

where  $\hat{\kappa}$  is a vector in the adjoint, orthogonal to  $\hat{\phi}$ . We can always write  $\hat{\kappa}$  as

$$\hat{\kappa} = h U \tau^2 U^\dagger h^\dagger , \quad (3.7)$$

where  $h$  and  $U$  are as defined before and in Eq. (2.8).

Using the identity  $\sigma_{ij}^\alpha \sigma_{kl}^\alpha = \delta_{il} \delta_{kj} - \frac{1}{2} \delta_{ij} \delta_{kl}$ , we find that  $\psi$  is a eigenvector of the expression on the left hand side of Eq. (3.6). Then writing the right hand side of that equation in terms of  $h$  and  $U$ , we find that  $\psi$  can be written as

$$\psi = v_2 h U \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} , \quad (3.8)$$

where  $\rho_1$  and  $\rho_2$  are constants. Keeping  $U$  fixed, we vary  $\xi$  and find the periodicity

$$\psi(\xi) = \psi(\xi + 4\pi) . \quad (3.9)$$

This  $\xi$  is the angle parameter of the residual  $U(1)$  gauge symmetry and in the presence of a string solution, this  $\xi$  is mapped a circle around the string. In order to make  $\psi$  single valued around the string, we need  $\xi = 2\chi$ , where  $\chi$  is the angular coordinate for a loop around the string. Next let us calculate the Lagrangian of the scalar field  $\psi$ . We have

$$D_\mu \psi = \partial_\mu \psi - ig A_\mu \psi \quad (3.10)$$

$$= \partial_\mu (h U \rho) - ig \left[ B_\mu \hat{\phi} + ig \left[ \hat{\phi}, \partial_\mu \hat{\phi} \right] \right] h U \rho \quad (3.11)$$

$$= \partial_\mu (U h_0 \rho) - ig \left[ B_\mu \hat{\phi} + ig \left[ \hat{\phi}, \partial_\mu \hat{\phi} \right] \right] U h_0 \rho \quad (3.12)$$

$$= -i U h_0 \tau^3 \rho [2\partial_\mu \chi + g (B_\mu + N_\mu)] , \quad (3.13)$$

where  $h_0 = e^{-i2\chi\tau^3}$ ,  $\rho^\dagger \rho = v_2^2$ , and we have used the identity  $U^\dagger h U = \exp(-2i\chi\tau^3)$ . We have also introduced the Abelian ‘monopole field’

$$N_\mu = 2i Q_m \text{Tr} \left[ \partial_\mu U U^\dagger \hat{\phi} \right] , \quad (3.14)$$

$$\partial_{[\mu} N_{\nu]} = Q_m M_{\mu\nu} + 2i Q_m \text{Tr} [(\partial_{[\mu} \partial_{\nu]} U) U^\dagger \hat{\phi}] . \quad (3.15)$$

The first term reproduces the magnetic field of the monopole configuration, while the second term is a gauge dependent line singularity, the Dirac string. This singular string is

a red herring, and we are going to ignore it because it is an artifact of our construction. We have used a  $U(\vec{x})$  which is appropriate for a point monopole. If we look at the system from far away, the monopoles will look like point objects and it would seem that we should find Dirac strings attached to each of them. However, we know that the 't Hooft-Polyakov monopoles are actually not point objects, and their near magnetic field is not describable by an Abelian four-potential  $N_\mu$ , so if we could do our calculations without the far-field approximation, we would not find a Dirac string. Further, as was pointed out in [12], the actual flux tube occurs along the line of vanishing  $\psi$ , and it is always possible to choose a  $U(\vec{x})$  appropriate for the monopole configuration such that the Dirac string lies along the zeroes of  $\psi$ . Since  $|\psi|^2$  always multiplies the term containing  $N_\mu$  in the action, the effect of the Dirac string can always be ignored.

With these definitions we can calculate

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{v_2^2}{2}(\partial_\mu\chi + e(B_\mu + N_\mu))^2 \quad (3.16)$$

Here, we have defined electric charge  $e = \frac{g}{2}$  and written the magnetic charge as  $Q_m = \frac{1}{2e}$ .

#### 4. Dualization

Let us now dualize the low energy effective action in order to express the theory in terms of the macroscopic string variables. The partition function  $Z$  is simply the functional integral

$$Z = \int \mathcal{D}B_\mu \mathcal{D}\chi \exp i \int d^4x \left[ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{v_2^2}{2}(eB_\mu + \partial_\mu\chi + eN_\mu)^2 \right]. \quad (4.1)$$

In the presence of flux tubes we can decompose the angle  $\chi$  into a part  $\chi^s$  which measures flux in the tube and a part  $\chi^r$  describing single valued fluctuations around this configuration,  $\chi = \chi^r + \chi^s$ . Then if  $\chi$  winds around the tube  $n$  times, we can define

$$\epsilon^{\mu\nu\rho\lambda}\partial_\rho\partial_\lambda\chi^s = 2\pi n \int_\Sigma d\sigma^{\mu\nu}(x(\xi))\delta^4(x - x(\xi)) \equiv \Sigma^{\mu\nu}, \quad (4.2)$$

where  $\xi = (\xi^1, \xi^2)$  are the coordinates on the world-sheet and  $d\sigma^{\mu\nu}(x(\xi)) = \epsilon^{ab}\partial_a x^\mu \partial_b x^\nu$ . The vorticity quantum is  $2\pi$  in the units we are using and  $n$  is the winding number [28].

The integration over  $\chi$  has now become integrations over both  $\chi^r$  and  $\chi^s$ . However  $\chi^r$  is a single-valued field, so it can be absorbed into the gauge field  $B_\mu$  by a redefinition, or gauge transformation,  $B_\mu \rightarrow B_\mu + \partial_\mu\chi^r$ . We can linearize the action by introducing auxiliary fields  $C_\mu, B_{\mu\nu}$  and  $A_\mu^m$ ,

$$Z = \int \mathcal{D}B_\mu \mathcal{D}C_\mu \mathcal{D}\chi_s \mathcal{D}B_{\mu\nu} \mathcal{D}A_\mu^m \exp i \int d^4x \left[ -\frac{1}{4}G^{\mu\nu}G_{\mu\nu} + \frac{1}{4}\epsilon^{\mu\nu\rho\lambda}G_{\mu\nu}F_{\rho\lambda} - \frac{1}{2v_2^2}C_\mu^2 - C^\mu(eB_\mu + eN_\mu + \partial_\mu\chi_s) \right], \quad (4.3)$$

where we have written  $G_{\mu\nu} = \partial_\mu A_\nu^m - \partial_\nu A_\mu^m + ev_2 B_{\mu\nu}$ . and  $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + M_{\mu\nu}$ . Now we can integrate over  $B_\mu$  easily,

$$Z = \int \mathcal{D}C_\mu \mathcal{D}\chi_s \mathcal{D}B_{\mu\nu} \mathcal{D}A_\mu^m \delta \left( C^\mu - \frac{v_2}{2} \epsilon^{\mu\nu\rho\lambda} \partial_\nu B_{\rho\lambda} \right) \exp i \int d^4x \left[ -\frac{1}{4} G^{\mu\nu} G_{\mu\nu} + \frac{ev_2}{4} \epsilon^{\mu\nu\rho\lambda} B_{\mu\nu} M_{\rho\lambda} - A^\mu j_\mu - \frac{1}{2v_2^2} C_\mu^2 - C^\mu (eB_\mu + eN_\mu + \partial_\mu \chi_s) \right]. \quad (4.4)$$

Here  $j_m^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\lambda} \partial_\nu M_{\rho\lambda}$  is the magnetic monopole current. Integrating over  $C_\mu$  we get

$$Z = \int \mathcal{D}\chi_s \mathcal{D}B_{\mu\nu} \mathcal{D}A_\mu^m \exp i \int d^4x \left[ -\frac{1}{4} G^{\mu\nu} G_{\mu\nu} + \frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho} - \frac{v_2}{2} \Sigma_{\mu\nu} B^{\mu\nu} - A^\mu j_\mu \right] \quad (4.5)$$

where we have written defined  $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ , used Eq. (4.2) and also written  $M_{\mu\nu} = (\partial_\mu N_\nu - \partial_\nu N_\mu)$ .

We can also replace the integration over  $\mathcal{D}\chi^s$  by an integration over  $\mathcal{D}x_\mu(\xi)$ , representing a sum over all the flux tube world sheet where  $x_\mu(\xi)$  parametrizes the surface of singularities of  $\chi$ . The Jacobian for this change of variables gives the action for the string on the background space time [19, 29]. The string has a dynamics given by the Nambu-Goto action, plus higher order operators [30], which can be obtained from the Jacobian. We will ignore the Jacobian below, but of course it is necessary to include it if we want to study the dynamics of the flux tube.

$$Z = \int \mathcal{D}x_\mu(\xi) \mathcal{D}B_{\mu\nu} \mathcal{D}A_\mu^m \exp i \int d^4x \left[ -\frac{1}{4} G^{\mu\nu} G_{\mu\nu} + \frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho} - \frac{v_2}{2} \Sigma_{\mu\nu} B^{\mu\nu} - A^\mu j_\mu \right] \quad (4.6)$$

The equations of motion for the field  $B_{\mu\nu}$  and  $A^\mu$  can be calculated from this to be

$$\partial_\lambda H^{\lambda\mu\nu} = -m G^{\mu\nu} - \frac{m}{e} \Sigma^{\mu\nu}, \quad (4.7)$$

$$\partial_\mu G^{\mu\nu} = j_m^\mu \quad (4.8)$$

where  $G_{\mu\nu} = ev_2 B_{\mu\nu} + \partial_\mu A_\nu^m - \partial_\nu A_\mu^m$ , and  $m = ev_2$ . Combining Eq. (4.8) and Eq. (4.7) we find that

$$\frac{1}{e} \partial_\mu \Sigma^{\mu\nu}(x) + j_m^\mu(x) = 0. \quad (4.9)$$

It follows rather obviously that a vanishing magnetic monopole current implies  $\partial_\mu \Sigma^{\mu\nu}(x) = 0$ , or in other words if there is no monopole in the system, the flux tubes will be closed.

The magnetic flux through the tube is  $\frac{2n\pi}{e}$ , while the total magnetic flux of the monopole is  $\frac{4m\pi}{g}$ , where  $n, m$  are integers. Since  $eQ_m = \frac{1}{2}$ , it follows that we can have a string that confine a monopole and anti-monopole pair for every integer  $n$ . Although this string configuration could be broken by creating a monopole-anti-monopole pair, there is a hierarchy of energy scales  $v_1 \gg v_2$ , which are respectively proportional to the mass of the monopole and the energy scale of the string. So this hierarchy can be expected to prevent string breakage by pair creation.



The conservation law of Eq. (4.9) also follows directly from  $Z$  in Eq. (4.6) by introducing a variable  $B'_{\mu\nu} = B_{\mu\nu} + \frac{1}{m}(\partial_\mu A_\nu^m - \partial_\nu A_\mu^m)$  and integrating over the field  $A_\mu^m$ . If we do so we get

$$Z = \int \mathcal{D}x_\mu(\xi) \mathcal{D}B'_{\mu\nu} \delta\left[\frac{1}{e}\partial_\mu \Sigma^{\mu\nu}(x) + j_m^\nu(x)\right] \exp\left[i \int \left\{ \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4} m^2 B_{\mu\nu}^{\prime 2} - \frac{m}{2e} \Sigma_{\mu\nu} B^{\prime\mu\nu} \right\}\right], \quad (4.10)$$

with the delta functional showing the conservation law (4.9). Thus these strings are analogous to the confining strings in three dimensions [31]. There is no  $A_\mu^m$ , the only gauge field which is present is  $B'_{\mu\nu}$ . This  $B'_{\mu\nu}$  field mediates the direct interaction between the confining strings.

The delta functional in Eq. (4.10) enforces that at every point of space-time, the monopole current cancels the currents of the end points of flux tube. So the monopole current must be non-zero only at the end of the flux tube. Eq. (4.10) does not carry Abelian gauge field  $A_\mu^m$ , only a massive second rank tensor gauge field. All this confirms the permanent attachment of monopoles at the end of the flux tube which does not allow gauge flux to escape out of the flux tubes. There are important differences between the results obtained from this construction and that from using two adjoint scalars. The mass of the Abelian photon will be zero for the two adjoint case if the two adjoint vevs are aligned in the same direction. But this cannot happen for one adjoint and one fundamental scalar. Also, in this case flux confinement is possible for all winding numbers of the string.

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